Remarks on the Jacobson radical

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For any ring R, we have:

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3. $\Delta(R)$ *is a subring of R;*
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5. *For any rings R_i, $i \in I$, $\Delta(\prod_{i \in I} R_i) = \prod_{i \in I} \Delta(R_i)$.*

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(4) If $\Delta(R)$ is an ideal of R and $r \in R$. Then for any $x \in \Delta(R)$, $rx + 1 \in U(R)$, and $\Delta(R) \subseteq J(R)$ follows, i.e. $\Delta(R) = J(R)$.

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Example

Let A be a commutative domain such that $J(A) \neq 0$ and define $R = A[t]$. Then $J(R) = 0$ and $U(R) = U(A)$, but for any $a \in J(A)$, we have

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So that $0 \neq J(A) \subseteq \Delta(R)$ but $J(R) = 0$.

Theorem

Let R be a unital ring and T be the subring of R generated by $U(R)$. Then:

1. $\Delta(R) = J(T)$ *and* $\Delta(S) = \Delta(R)$*, for any subring S of R such that $T \subseteq S$;*
2. $\Delta(R)$ *is the largest Jacobson radical ring contained in R which is closed with respect to multiplication by units of R.*

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Corollary

Let R be a ring such that every element of R is a sum of units. Then $\Delta(R) = J(R)$.

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1. KG where K is a division ring, G a group.
2. $M_n(R)$ for $n \geq 2$ and R is any ring.
3. A clean ring R such that $2 \in U(R)$.
4. $K[x, x^{-1}; \sigma]$ where $\sigma \in Aut(K)$.

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5. $\Delta(R) \cap Z(R) \subseteq \Delta(Z(R))$.

An example

Let A be a commutative domain with $J(A) \neq 0$ and $S = A[x]$. Then $0 \neq J(A) \subseteq \Delta(S)$ and clearly $J(S) = 0$. $R = M_2(S)$, where A is a commutative local domain. As we have seen $\Delta(R) = J(R) = 0$. Notice that the center $Z = Z(R)$ of $R = M_2(S)$ is isomorphic to S and $U(Z) = U(R) \cap Z$. Therefore $0 = \Delta(R) \cap Z \subseteq \Delta(Z) \simeq J(A) \neq 0$. Thus the inclusion from the above corollary (5) can be strict even when $J(R) = 0 = J(Z(R))$.

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For a ring R ($1 \in R$) and a subring $S \subset R$ such that $1 \notin S$, we denote $\widehat{S}$ the subring of R generated by $S \cup \{1\}$.

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Examples

For any ring R denote $T_n(R)$ the upper triangular matrix rings over R and $J_n(R)$ all strictly upper triangular matrices.

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5. *R has stable range 1.*
6. *$R = FG$ is a group algebra over a field F.*

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(3) Let $e^2 = e$ be such that $e\Delta(R)e \subseteq \Delta(R)$. Then $e\Delta(R)e \subseteq \Delta(eRe)$.

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Examples

(1)Let S be any ring such that $J(S) = 0$ and $\Delta(S) \neq 0$ and let $R = M_2(S)$. Then $\Delta(R) = J(R) = 0$. Therefore, if $e = e_{11} \in R$, then $e\Delta(R)e = eJ(R)e = J(eRe) = 0$. and $\Delta(eRe) \simeq \Delta(S) \neq 0$. This shows that the inclusion $e\Delta(R)e \subsetneq \Delta(eRe)$.

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(2) Let $R = \mathbb{F}_2 < x, y > / < x^2 >$. Then $J(R) = 0$ and $U(R) = 1 + \mathbb{F}_2 x + xRx$. In particular $\mathbb{F}_2 x + xRx$ is contained in $\Delta(R)$ but $J(R) = 0$.

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Let us end this part with the following proposition.

Proposition

Let R be 2-primal ring. Then $\Delta(R[x]) = \Delta(R) + J(R[x])$.

Another Short Story

Quasi-inverses and reflexive inverses

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An element $a \in R$ is a (von Neumann) regular element if $I(a) := \{x \in R \mid a = axa\} \neq \emptyset$. In this case $ref(a) = \{x \in I(a) \mid xax = x\} \neq \emptyset$. The element of $ref(a)$ are the reflexive inverses of a.

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Lemma

For $a \in R$ and $a_0 \in I(a)$, we have

$$I(a) = \{a_0 + t - a_0ataa_0 \mid t \in R\}.$$

Proposition

For $a \in Reg(R)$, *let* $\varphi_a : I(a) \longrightarrow Ref(a)$ *be such that* $\varphi_a(x) = xax$. *Then*

1. *The map* φ_a *is onto.*
2. $Ref(a) = I(a)aI(a)$.
3. *If* $x, y \in I(a)$ *are such that* $\varphi_a(x) = \varphi_a(y)$ *then* $x - y \in l(a) \cap r(a)$.
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4. *Let $x \in Ref(a)$, then $\varphi_a(x) = x$.*

Lemma

Let $a_0 \in I(a)$. Write $e = aa_0$, $f = a_0a$ and $e' = 1 - e$, $f' = 1 - f$. Then

(i) $lann(a) = l(a) + r(a) = Re' + f'R$.

(ii) $I(a) = a_0 + lann(a) = a_0 + Re' + f'R$.

(iii) *If $a_0 \in Ref(a)$, then $Ref(a) = a_0 + fRe' + f'Re + f'RaRe'$.*

Proposition

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Theorem

Let a, b be two elements in a semiprime ring R, the following are equivalent

1. $a = b$
2. $Ref(a) = Ref(b)$
3. $I(a) = I(b)$

THANK YOU !